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RELATIONSHIP BETWEEN MIRROR GIMBAL ANGLES AND STEREO AND OBLIQUITY ANGLES

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In this report, relationships between the mirror gimbal angles and the photographic or line-of-sight angles are derived for three coordinate systems: S_v, S_r and S_r'.

S_v Coordinate System

The photographic angles are defined in the vehicle coordinate system, S_v, as a roll about the x_v axis (Ω_v), then a pitch about the displaced y_v axis (Σ_v). The definition of these angles in terms of main tracking mirror gimbal angles and the ATS tracking mirror gimbal angles follows.

Stereo and Obliquity Angles:

Rotations as shown in Figure 1:

$S_v \to S_1$ $\quad\quad$ Ω_v about x_v

$S_1 \to S_2$ $\quad\quad$ Σ_v about y_1

The transformation equations are:

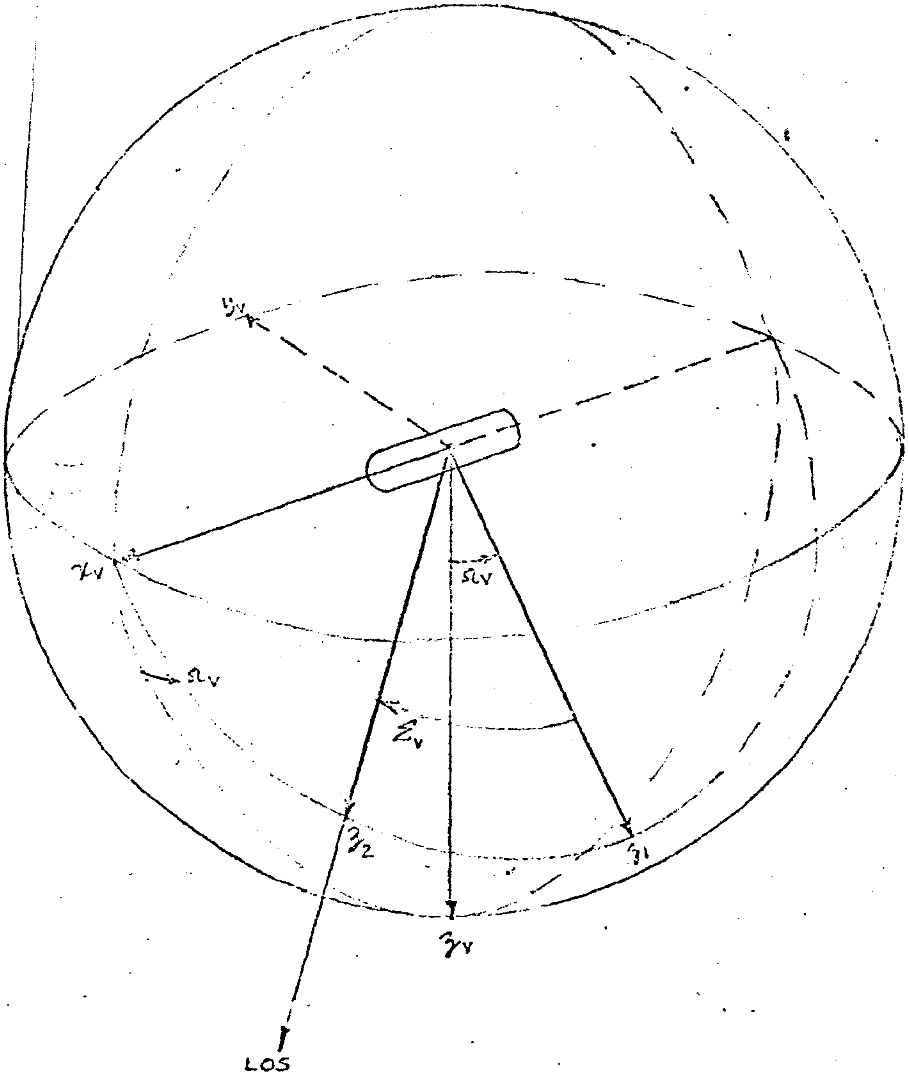
$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} c\Sigma_v & 0 & -s\Sigma_v \\ 0 & 1 & 0 \\ s\Sigma_v & 0 & c\Sigma_v \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\Omega_v & s\Omega_v \\ 0 & -s\Omega_v & c\Omega_v \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} c\Sigma_v & s\Sigma_v\, s\Omega_v & -s\Sigma_v\, c\Omega_v \\ 0 & c\Omega_v & s\Omega_v \\ s\Sigma_v & -c\Sigma_v\, s\Omega_v & c\Sigma_v\, c\Omega_v \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \quad (2)$$

where

$s = \sin$

$c = \cos$



STEREO. AND OBLIQUITY ANGLES

FIGURE 1

Main Tracking Mirror Gimbal Angles:

The angle φ is a roll about the main tracking mirror roll gimbal and θ is a pitch about the main tracking mirror pitch gimbal. The angle θ_1 is the -2 degree pitch rotation from the S_v to the S_b coordinate system.

Rotations:

$S_v \rightarrow S_b$	θ_1 about the y_v
$S_b \rightarrow S_m$	φ about x_b
$S_m \rightarrow S_m'$	2θ about y_m

The rotations and the rotations for stereo and obliquity are shown in Figure 2.

The transformation equations are:

$$S_m' = \left[2\theta\right]_{y_m} \left[\varphi\right]_{x_b} \left[\theta_1\right]_{y_v} S_v \tag{3}$$

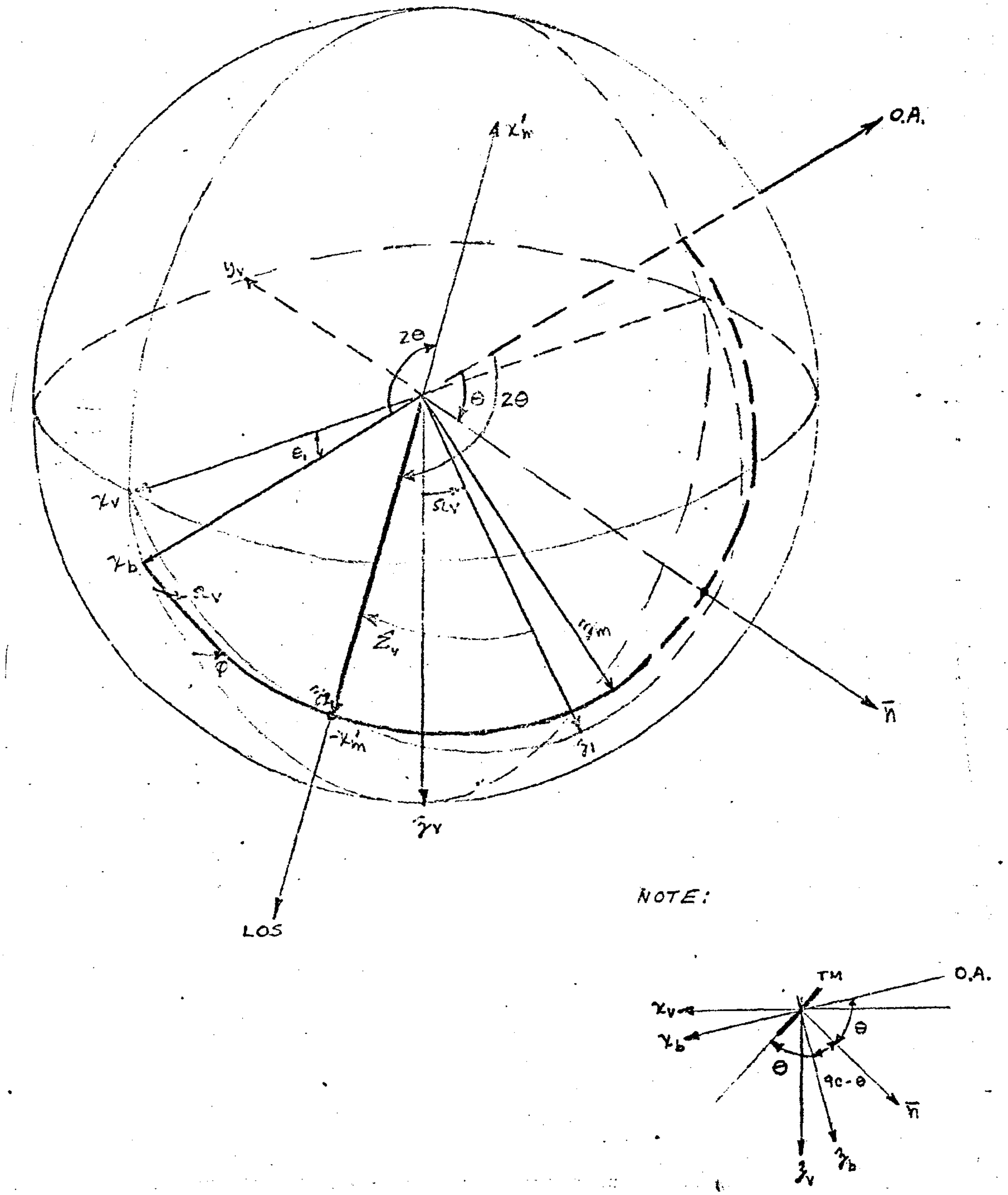
or

$$\begin{bmatrix} x_m' \\ y_m' \\ z_m' \end{bmatrix} = \begin{bmatrix} c2\theta & 0 & -s2\theta \\ 0 & 1 & 0 \\ s2\theta & 0 & c2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & s\varphi \\ 0 & -s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 \\ 0 & 1 & 0 \\ s\theta_1 & 0 & c\theta_1 \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \tag{4}$$

where $\theta_1 = -2$ deg.

which yields

$$\begin{bmatrix} x_m' \\ y_m' \\ z_m' \end{bmatrix} = \begin{bmatrix} c2\theta\, c\theta_1 - s2\theta\, c\varphi\, s\theta_1 & s2\theta s\varphi & -c2\theta s\theta_1 - s2\theta c\varphi c\theta_1 \\ s\varphi s\theta_1 & c\varphi & s\varphi c\theta_1 \\ s2\theta\, c\theta_1 + c2\theta\, c\varphi\, s\theta_1 & -c2\theta\, s\varphi & -s2\theta\, s\theta_1 + c2\theta\, c\varphi\, c\theta_1 \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \tag{5}$$



MAIN TRACKING MIRROR GIMBAL ANGLES

FIGURE 2

Since z_2 coincides with $-x_m'$

$$s\Sigma_v = s2\theta\, c\varphi\, s\theta_1 - c2\theta\, c\theta_1 \tag{6}$$

$$c\Sigma_v\, s\Omega_v = s2\theta\, s\varphi \tag{7}$$

$$c\Sigma_v\, c\Omega_v = + c2\theta\, s\theta_1 + s2\theta\, c\varphi\, c\theta_1 \tag{8}$$

From Equation (7)

$$s\Omega_v = \frac{s2\theta\, s\varphi}{c\Sigma_v} \tag{9}$$

Note from Equation (6) that for $\Sigma_v = 0$ with $\Omega_v = 0$

$$\tan 2\theta = \cot \theta_1 \tag{10}$$

Since $\theta_1 = -2$ deg, $\theta = 46^\circ$.

Introducing $\Delta\theta$, which is the angular deviation from the $\Sigma_v = 0$ mirror position, into Equations (6) and (9) yields:

$$\theta = 46^\circ + \Delta\theta \tag{11}$$

$$\boxed{s\Sigma_v = s2\Delta\theta(c\varphi\, s^2\theta_1 + c^2\theta_1) + c2\Delta\theta\, s\theta_1 c\theta_1(c\varphi - 1)} \tag{12}$$

$$\boxed{s\Omega_v = \frac{s\varphi}{c\Sigma_v}\,(s2\Delta\theta\, s\theta_1 + c2\Delta\theta\, c\theta_1)} \tag{13}$$

The following expressions to obtain gimbal angles given Σ_v and Ω_v can be derived using Equations (6) through (8).

$$\boxed{c2\theta = c\Sigma_v\, c\Omega_v\, s\theta_1 - s\Sigma_v\, c\theta_1} \tag{14}$$

where $0 \le \theta \le +90$ deg

$$s\varphi = \frac{c\Sigma_v \, s\Omega_v}{s2\theta} \qquad (15)$$

$$\Delta\theta = \theta - 46 \text{ deg} \qquad (16)$$

ATS Tracking Mirror Gimbal Angles:

The angle φ_a is a roll about the ATS tracking mirror roll gimbal and θ_a is a pitch about the ATS tracking mirror pitch gimbal. The angle ψ_2 is the 9 degree yaw rotation from the S_v to the S_a coordinate system.

Rotations:

S_v	$\rightarrow S_a$	ψ_2 about z_v
S_a	$\rightarrow S_{ma}$	φ_a about x_a
S_{ma}	$\rightarrow S_{ma}'$	θ_a about y_{ma}

The above rotations and the rotations for stereo and obliquity are shown in Figure 3.

The transformation equations are:

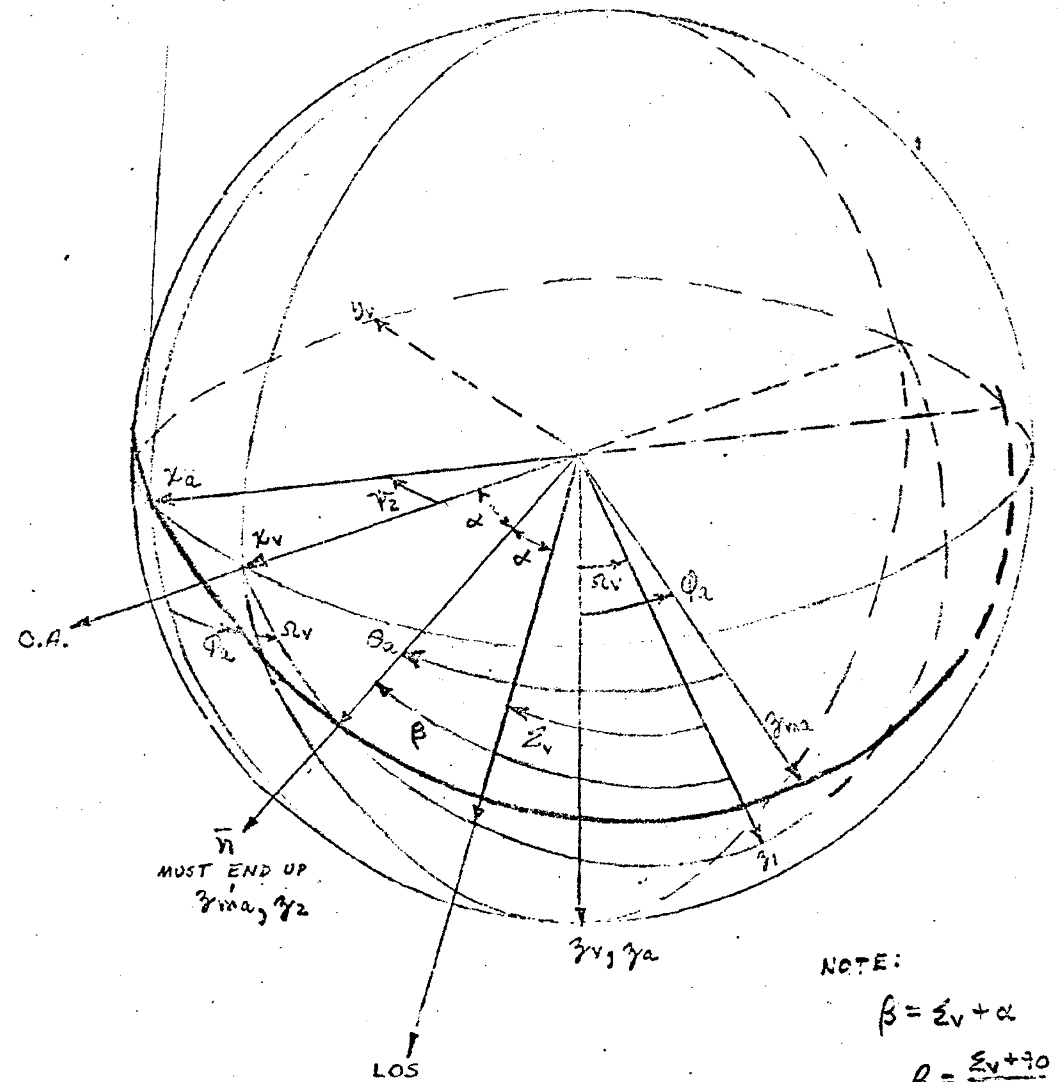
$$S_{ma}' = [\theta_a]_{y_{ma}} [\varphi_a]_{x_a} [\psi_2]_{z_v} S_v \tag{17}$$

or

$$\begin{bmatrix} x_{ma}' \\ y_{ma}' \\ z_{ma} \end{bmatrix} = \begin{bmatrix} c\theta_a & 0 & -s\theta_a \\ 0 & 1 & 0 \\ s\theta_a & 0 & c\theta_a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi_a & s\varphi_a \\ 0 & -s\varphi_a & c\varphi_a \end{bmatrix} \begin{bmatrix} c\psi_2 & s\psi_2 & 0 \\ -s\psi_2 & c\psi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \tag{18}$$

where $\psi_2 = 9$ deg, which yields

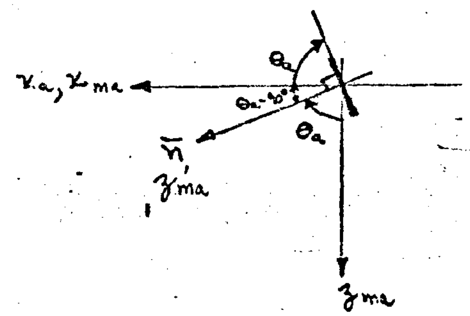
$$\begin{bmatrix} x_{ma}' \\ y_{ma}' \\ z_{ma}' \end{bmatrix} = \begin{bmatrix} c\theta_a c\psi_2 - s\theta_a s\varphi_a s\psi_2 & c\theta_a s\psi_2 + s\theta_a s\varphi_a c\psi_2 & -s\theta_a c\varphi_a \\ -c\varphi_a s\psi_2 & c\varphi_a c\psi_2 & s\varphi_a \\ s\theta_a c\psi_2 + c\theta_a s\varphi_a s\psi_2 & s\theta_a s\psi_2 - c\theta_a s\varphi_a c\psi_2 & c\theta_a c\varphi_a \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \tag{19}$$



NOTE:

$\beta = \Sigma_v + \alpha \qquad 2\alpha + \Sigma_v = 90°$

$\beta = \frac{\Sigma_v + 90}{2}$



ATS TRACKING MIRROR GIMBAL ANGLES

FIGURE 3

To relate Σ_v and Ω_v to the mirror gimbal angles, the mirror normal will be placed in the plane containing and with an equal angle between the optical axis and the line-of-sight (see Figure 3). The rotations shown in Figure 3 are:

$S_v \rightarrow S_1$ Ω_v about x_v

$S_1 \rightarrow S_2$ β about y_1

where

$$\beta = \frac{\Sigma_v + 90^o}{2}$$

The transformation equations are:

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} c\beta & s\beta\, s\Omega_v & -s\beta\, c\Omega_v \\ 0 & c\Omega_v & s\Omega_v \\ s\beta & -c\beta\, s\Omega_v & c\beta\, c\Omega_v \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \quad (20)$$

Since z_2 conicides with z_{ma}'

$$\boxed{s\beta = s\theta_a\, c\psi_2 + c\theta_a\, s\varphi_a\, s\psi_2} \quad (21)$$

$$-c\beta\, s\Omega_v = s\theta_a\, s\psi_2 - c\theta_a\, s\varphi_a\, c\psi_2 \quad (22)$$

$$c\beta\, c\Omega_v = c\theta_a\, c\varphi_a \quad (23)$$

where

$$\boxed{\Sigma_v = 2\beta - 90 \text{ deg}} \quad (24)$$

Using Equations (21) and (22) yields:

$$\boxed{s\Omega_v = \frac{s\beta\, c\psi_2 - s\theta_a}{c\beta\, s\psi_2}} \quad (25)$$

Gimbal angles are found in terms of Σ_v and Ω_v from Equations (21) through (24) as:

$$\boxed{\beta = \frac{\Sigma_v + 90^\circ}{2}} \qquad (26)$$

$$\boxed{s\theta_a = s\beta c\psi_2 - c\beta s\Omega_v s\psi_2} \qquad (27)$$

$$\boxed{s\varphi_a = \frac{s\beta s\psi_2 + c\beta s\Omega_v c\psi_2}{c\theta_a}} \qquad (28)$$

Note from these equations that when

$\Sigma_v = 0$ and $\Omega_v = 0$,

$\theta_a = 44.299$ deg and

$\varphi_a = 8.891$ deg.

$\underline{S_r}$ Coordinate System

If the stereo and obliquity angles are to be expressed in the LV/RVV coordinate system, S_r, the following relations exist between the tracking mirror gimbal angles and the stereo and obliquity angles.

Stereo and Obliquity Angles:

Rotations:

$S_r \rightarrow S_1$ $\quad\quad \Omega_r$ about x_r

$S_1 \rightarrow S_2$ $\quad\quad \Sigma_r$ about y_1

From previous relations,

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} c\Sigma_r & s\Sigma_r\, s\Omega_r & -s\Sigma_r\, c\Omega_r \\ 0 & c\Omega_r & s\Omega_r \\ s\Sigma_r & -c\Sigma_r\, s\Omega_r & c\Sigma_r\, c\Omega_r \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} \quad (29)$$

where the unit line-of-sight vector, $\hat{k}$, is $\hat{\underline{z}}_2$.

The attitude matrix, A, relates S_r to S_v, i.e.,

$$A: S_r \rightarrow S_v$$

is

$$A = \left[\theta_v\right]_y \left[\varphi_v\right]_x \left[\psi_v\right]_z \quad (30)$$

where the order of rotation is not important since small angles will be assumed. φ_v, θ_v, and ψ_v are respectively, the roll, pitch and yaw angles of the body.

$$A = \begin{bmatrix} 1 & 0 & -\theta_v \\ 0 & 1 & 0 \\ \theta_v & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varphi_v \\ 0 & -\varphi_v & 1 \end{bmatrix} \begin{bmatrix} 1 & \psi_v & 0 \\ -\psi_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (31)$$

Assuming multiples of small angles equals zero

$$\begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} = \begin{bmatrix} 1 & \psi_v & -\theta_v \\ -\psi_v & 1 & \varphi_v \\ \theta_v & -\varphi_v & 1 \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} \tag{32}$$

From previous results

$$\begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} = \begin{bmatrix} c\Sigma_v & s\Sigma_v s\Omega_v & -s\Sigma_v c\Omega_v \\ 0 & c\Omega_v & s\Omega_v \\ s\Sigma_v & -c\Sigma_v s\Omega_v & c\Sigma_v c\Omega_v \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \tag{33}$$

where $\hat{z}_2'$ is the unit LOS vector $\hat{k}'$.

Combining Equations (32) and (33) yields:

$$\begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} = \begin{bmatrix} c\Sigma_v & s\Sigma_v s\Omega_v & -s\Sigma_v c\Omega_v \\ 0 & c\Omega_v & s\Omega_v \\ s\Sigma_v & -c\Sigma_v s\Omega_v & c\Sigma_v c\Omega_v \end{bmatrix} \begin{bmatrix} 1 & \psi_v & -\theta_v \\ -\psi_v & 1 & \varphi_v \\ \theta_v & -\varphi_v & 1 \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} = \begin{bmatrix} \left(c\Sigma_v - s\Sigma_v s\Omega_v \psi_v - s\Sigma_v c\Omega_v \theta_v\right) & \left(c\Sigma_v \psi_v + s\Sigma_v s\Omega_v + s\Sigma_v c\Omega_v \varphi_v\right) & \left(-c\Sigma_v \theta_v + s\Sigma_v s\Omega_v \varphi_v - s\Sigma_v c\Omega_v\right) \\ \left(-c\Omega_v \psi_v + s\Omega_v \theta_v\right) & \left(c\Omega_v - s\Omega_v \varphi_v\right) & \left(c\Omega_v \varphi_v + s\Omega_v\right) \\ \left(s\Sigma_v + c\Sigma_v s\Omega_v \psi_v + c\Sigma_v c\Omega_v \theta_v\right) & \left(s\Sigma_v \psi_v - c\Sigma_v s\Omega_v - c\Sigma_v c\Omega_v \varphi_v\right) & \left(-s\Sigma_v \theta_v - c\Sigma_v s\Omega_v \varphi_v + c\Sigma_v c\Omega_v\right) \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} \tag{35}$$

Since z_2' coincides with z_2

$$s\Sigma_r = s\Sigma_v + c\Sigma_v s\Omega_v \psi_v + c\Sigma_v c\Omega_v \theta_v \tag{36}$$

$$-c\Sigma_r s\Omega_r = s\Sigma_v \psi_v - c\Sigma_v s\Omega_v - c\Sigma_v c\Omega_v \varphi_v \tag{37}$$

$$c\Sigma_r c\Omega_r = -s\Sigma_v \theta_v - c\Sigma_v s\Omega_v \varphi_v + c\Sigma_v c\Omega_v \tag{38}$$

Therefore, from Equation (36),

$$\boxed{s\Sigma_r = s\Sigma_v + c\Sigma_v\,(s\Omega_v\psi_v + c\Omega_v\theta_v)} \qquad (39)$$

and from Equation (37),

$$\boxed{s\Omega_r = \frac{c\Sigma_v}{c\Sigma_r}\,(s\Omega_v + c\Omega_v\varphi_v) - \frac{s\Sigma_v}{c\Sigma_r}\,\psi_v} \qquad (40)$$

Multiplying Equation (36) by φ_v and (37) by θ_v, then summing yields:

$$s\Sigma_r\varphi_r - c\Sigma_r s\Omega_r\theta_v = s\Sigma_v(\varphi_v + \psi_v\theta_v) + c\Sigma_v s\Omega_v\,(\psi_v\varphi_v - \theta_v) \qquad (41)$$

Multiplying Equation (38) by φ_v and summing with Equation (37) yields:

$$-c\Sigma_r s\Omega_r + c\Sigma_r c\Omega_r\varphi_v = s\Sigma_v(\psi_v - \theta_v\varphi_v) - c\Sigma_v s\Omega_v(1 + \varphi_v^2) \qquad (42)$$

Multiplying Equation (42) by $(\psi_v\varphi_v - \theta_v)$ and (41) by $(1 + \varphi_v^2)$, then summing yields:

$$s\Sigma_v\left[1 + \cancelto{0}{\varphi_v^2} + \cancelto{0}{\psi_v^2} + \cancelto{0}{\theta_v^2}\right] = s\Sigma_r\left[1 + \cancelto{0}{\varphi_v^2}\right] - c\Sigma_r\left[s\Omega_r(\psi_v + \cancelto{0}{\theta_v\varphi_v}) + c\Omega_r(\theta_v - \cancelto{0}{\psi_v\varphi_v})\right] \qquad (43)$$

Equation (43) can be reduced to

$$\boxed{s\Sigma_v = s\Sigma_r - c\Sigma_r\,(s\Omega_r\psi_v + c\Omega_r\theta_v)} \qquad (44)$$

Multiplying Equation (41) by $(\psi_v - \theta_v\varphi_v)$ and (42) by $-(\varphi_v + \psi_v\theta_v)$, then summing yields:

$$c\Sigma_v s\Omega_v\left[1 + \cancelto{0}{\varphi_v^2} + \cancelto{0}{\psi_v^2} + \cancelto{0}{\theta_v^2}\right] = s\Sigma_r\left[\psi_v - \cancelto{0}{\theta_v\varphi_v}\right] - c\Sigma_r\left[-s\Omega_r(1 + \cancelto{0}{\theta_v^2}) + c\Omega_r(\varphi_v + \cancelto{0}{\theta_v\psi_v})\right] \qquad (45)$$

Equation (45) can be reduced to

$$\boxed{s\Omega_v = \frac{c\Sigma_r}{c\Sigma_v}(s\Omega_r - c\Omega_r\varphi_v) + \frac{s\Sigma_r}{c\Sigma_v}\,\psi_v} \qquad (46)$$

$S_r{}'$ Coordinate System

If the stereo and obliquity angles are to be expressed in the LV/OP coordinate system, $S_r{}'$, the following relations exist between the tracking mirror gimbal angles and the stereo and obliquity angles.

Stereo and Obliquity Angles

Rotations:

$S_r{}' \rightarrow S_1$ $\Omega_r{}'$ about $x_r{}'$

$S_1 \rightarrow S_2$ $\Sigma_r{}'$ about y_1

From previous relations

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} c\Sigma_r{}' & s\Sigma_r{}' s\Omega_r{}' & -s\Sigma_r{}' c\Omega_r{}' \\ 0 & c\Omega_r{}' & s\Omega_r{}' \\ s\Sigma_r{}' & -c\Sigma_r{}' s\Omega_r{}' & c\Sigma_r{}' c\Omega_r{}' \end{bmatrix} \begin{bmatrix} x_r{}' \\ y_r{}' \\ z_r{}' \end{bmatrix} \tag{47}$$

where the unit LOS Vector, $\hat{\underline{k}}$, is $\hat{\underline{z}}_2$.

Assume the crab angle, yaw angle between the orbit plane and the relative velocity vector, is defined as η. Then

$S_r{}' \rightarrow S_r$ η about $z_r{}'$

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} c\eta & s\eta & 0 \\ -s\eta & c\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r{}' \\ y_r{}' \\ z_r{}' \end{bmatrix} \tag{48}$$

From Equation (35) assuming η is a small angle

$$\begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} = \begin{bmatrix} (c\Sigma_v - s\Sigma_v s\Omega_v \psi_v - s\Sigma_v c\Omega_v \theta_v) & (c\Sigma_v \psi_v + s\Sigma_v s\Omega_v + s\Sigma_v c\Omega_v \varphi_v) & (-c\Sigma_v \theta_v + s\Sigma_v s\Omega_v \varphi_v - s\Sigma_v c\Omega_v) \\ (-c\Omega_v \psi_v + s\Omega_v \theta_v) & (c\Omega_v - s\Omega_v \varphi_v) & (c\Omega_v \varphi_v + s\Omega_v) \\ (s\Sigma_v + c\Sigma_v s\Omega_v \psi_v + c\Sigma_v c\Omega_v \theta_v) & (s\Sigma_v \psi_v - c\Sigma_v s\Omega_v - c\Sigma_v c\Omega_v \varphi_v) & (-s\Sigma_v \theta_v - c\Sigma_v s\Omega_v \varphi_v + c\Sigma_v c\Omega_v) \end{bmatrix} \begin{bmatrix} 1 & \eta & 0 \\ -\eta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r' \\ y_r' \\ z_r' \end{bmatrix} \tag{49}$$

where $\hat{\underline{z}}_2{}'$ is the unit LOS vector $\hat{\underline{k}}$.

Since z_2' coincides with z_2

$$s\Sigma_r' = s\Sigma_v[1 - \psi_v \eta] + c\Sigma_v[s\Omega_v(\psi_v + \eta) + c\Omega_v(\theta_v + \varphi_v \eta)] \tag{50}$$

$$-c\Sigma_r' s\Omega_r' = s\Sigma_v[\psi_v + \eta] - c\Sigma_v[+ s\Omega_v (1 - \psi_v \eta) + c\Omega_v (\varphi_v - \theta_v \eta)] \tag{51}$$

$$c\Sigma_r' c\Omega_r' = -s\Sigma_v \theta_v - c\Sigma_v [s\Omega_v \varphi_v - c\Omega_v] \tag{52}$$

Therefore, from Equation (A-42),

$$\boxed{s\Sigma_r' = s\Sigma_v + c\Sigma_v [s\Omega_v (\psi_v + \eta) + c\Omega_v \theta_v]} \tag{53}$$

and from Equation (A-43),

$$\boxed{s\Omega_r' = \frac{c\Sigma_v}{c\Sigma_r'} (s\Omega_v + c\Omega_v \varphi_v) - \frac{s\Sigma_v}{c\Sigma_r'} (\psi_v + \eta)} \tag{54}$$

In order to simplify the following expressions, let

$$A = s\Sigma_r' \qquad G = \theta_v + \varphi_v \eta$$

$$B = -c\Sigma_r' s\Omega_r' \qquad H = \varphi_v - \theta_v \eta$$

$$D = c\Sigma_r' c\Omega_r' \qquad J = \theta_v$$

$$E = 1 - \psi_v \eta \qquad K = \varphi_v$$

$$F = \psi_v + \eta$$

Substituting these expressions into Equations (50) through (52) yields:

$$A = s\Sigma_r E + c\Sigma_v s\Omega_v F + c\Sigma_v c\Omega_v G \qquad (55)$$

$$B = s\Sigma_v F - c\Sigma_v s\Omega_v E - c\Sigma_v c\Omega_v H \qquad (56)$$

$$D = -s\Sigma_v J - c\Sigma_v s\Omega_v K + c\Sigma_v c\Omega_v \qquad (57)$$

Multiplying Equation (57) by H and summing with Equation (56) yields:

$$B + DH = s\Sigma_v (F - JH) - c\Sigma_v s\Omega_v (E + KH) \qquad (58)$$

Multiplying Equation (55) by H and (56) by G, then summing yields:

$$AH + BG = s\Sigma_v (EH + FG) + c\Sigma_v s\Omega_v (FH-EG) \qquad (59)$$

Multiplying Equation (58) by (FH-EG) and (59) by (E + KH), then summing yields:

$$s\Sigma_v [F^2 - FHJ + EGJ + E^2 + EHK + FGK] = A(E+HK) + B(F+GK) + D(-EG+FH) \qquad (60)$$

Substituting for the dummy variables yields:

$$s\Sigma_v[1 + \underbrace{\psi_v^2}_{0} + \underbrace{\theta_v^2}_{0} + \underbrace{\varphi_v^2}_{0} + \underbrace{\eta^2}_{0}(1 + \underbrace{\psi_v^2}_{0} + \underbrace{\theta_v^2}_{0} + \underbrace{\varphi_v^2}_{0})] = A[1 - \underbrace{\psi_v \eta}_{0} + \underbrace{\varphi_v^2}_{0} - \underbrace{\varphi_v \theta_v \eta}_{0}]$$
$$+ B[\psi_v + \eta + \underbrace{\theta_v \varphi_v}_{0} + \underbrace{\varphi_v^2 \eta}_{0}] + D[-\theta_v - \underbrace{\theta_v \eta^2}_{0} + \underbrace{\varphi_v \psi_v}_{0} + \underbrace{\varphi_v \psi_v \eta^2}_{0}] \qquad (61)$$

Equation (61) can be reduced to:

$$\boxed{s\Sigma_v = s\Sigma_r{}' - c\Sigma_r{}' [s\Omega_r{}' (\psi_v + \eta) + c\Omega_r{}' \theta_v]} \qquad (62)$$

Multiplying Equation (58) by (EH + FG) and (59) by -(F - JH), then summing yields:

$$-c\Sigma_v s\Omega_v [E^2 + F^2 + E(HK + GJ) + F(GK - HJ)] = A[HJ - F] + B[GJ + E] + D[EH + FG] \qquad (63)$$

Substituting for the dummy variables yields:

$$-c\Sigma_v s\Omega_v [1 + \cancelto{0}{\psi_v^2} + \cancelto{0}{\theta_v^2} + \cancelto{0}{\varphi_v^2} + \cancelto{0}{\eta^2} (1 + \cancelto{0}{\psi_v^2} + \cancelto{0}{\theta_v^2} + \cancelto{0}{\varphi_v^2})] =$$
$$A[-\psi_v - \eta + \cancelto{0}{\theta_v \varphi_v} - \cancelto{0}{\theta_v^2 \eta}] + B[1 - \cancelto{0}{\psi_v \eta} + \cancelto{0}{\theta_v \varphi_v \eta} + \cancelto{0}{\theta_v^2}] + D[\varphi_v + \cancelto{0}{\theta_v \psi_v}$$
$$+ \cancelto{0}{\varphi_v \eta^2} + \cancelto{0}{\theta_v \psi_v \eta^2}] \qquad (64)$$

Equation (64) reduces to

$$\boxed{s\Omega_v = \frac{c\Sigma_r'}{c\Sigma_v} (s\Omega_r' - c\Omega_r' \varphi_v) + \frac{s\Sigma_r'}{c\Sigma_v} (\psi_v + \eta)} \qquad (65)$$

SUMMARY

The expressions derived in this report to go between all cordinate systems are summarized below.

S_v Coordinate System

Main Tracking Mirror

$$s\Sigma_v = s2\Delta\theta(c\varphi s^2\theta_1 + c^2\theta_1) + c2\Delta\theta\, s\theta_1\, c\theta_1(c\varphi - 1) \tag{66}$$

$$s\Omega_v = \frac{s\varphi}{c\Sigma_v}(s2\Delta\theta\, s\theta_1 + c2\Delta\theta\, c\theta_1) \tag{67}$$

$$c2\theta = c\Sigma_v\, c\Omega_v\, s\theta_1 - s\Sigma_v c\theta_1 \text{ where } 0 \le \theta \le +90 \text{ deg} \tag{68}$$

$$s\varphi = \frac{c\Sigma_v\, s\Omega_v}{s2\theta} \tag{69}$$

$$\Delta\theta = \theta - 46 \text{ deg} \tag{70}$$

where $\theta_1 = -2$ deg

ATS Tracking Mirror

$$s\beta = s\theta_a\, c\psi_2 + c\theta_a\, s\varphi_a\, s\psi_2 \tag{71}$$

$$\Sigma_v = 2\beta - 90 \text{ deg} \tag{72}$$

$$s\Omega_v = \frac{1}{c\beta\, s\psi_2}(s\beta\, c\psi_2 - s\theta_a) \tag{73}$$

$$\beta = \frac{1}{2}(\Sigma_v + 90 \text{ deg}) \tag{74}$$

$$s\theta_a = s\beta\, c\psi_2 - c\beta\, s\Omega_v\, s\psi_2 \tag{75}$$

$$s\varphi_a = \frac{1}{c\theta_a}(s\beta\, s\psi_2 + c\beta\, s\Omega_v\, c\psi_2) \tag{76}$$

where $\psi_2 = 9$ deg.

S_r Coordinate System

$$s\Sigma_r = s\Sigma_v + c\Sigma_v (s\Omega_v \psi_v + c\Omega_v \theta_v) \tag{77}$$

$$s\Omega_r = \frac{c\Sigma_v}{c\Sigma_r} (s\Omega_v + c\Omega_v \varphi_v) - \frac{s\Sigma_v}{c\Sigma_r} \psi_v \tag{78}$$

$$s\Sigma_v = s\Sigma_r - c\Sigma_r (s\Omega_r \psi_v + c\Omega_r \theta_v) \tag{79}$$

$$s\Omega_v = \frac{c\Sigma_r}{c\Sigma_v} (s\Omega_r - c\Omega_r \varphi_v) + \frac{s\Sigma_r}{c\Sigma_v} \psi_v \tag{80}$$

S_r' Coordinate System

$$s\Sigma_r' = s\Sigma_v + c\Sigma_v [s\Omega_v (\psi_v + \eta) + c\Omega_v \theta_v] \tag{81}$$

$$s\Omega_r' = \frac{c\Sigma_v}{c\Sigma_r'} (s\Omega_v + c\Omega_v \varphi_v) - \frac{s\Sigma_v}{c\Sigma_r'} (\psi_v + \eta) \tag{82}$$

$$s\Sigma_v = s\Sigma_r' - c\Sigma_r' [s\Omega_r' (\psi_v + \eta) + c\Omega_r' \theta_v] \tag{83}$$

$$s\Omega_v = \frac{c\Sigma_r'}{c\Sigma_v} (s\Omega_r' - c\Omega_r' \varphi_v) + \frac{s\Sigma_r'}{c\Sigma_v} (\psi_v + \eta) \tag{84}$$